数理解析研究所講究録
1158 巻 2000 年 87-97

Computing point residues for a shape basis case via differential operators

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1 Introduction

In this paper, we study computational aspects of point residues. We concentrate on a shape basis case and we present algorithms which compute point residues for this generic case.

In 1987, Gianni and Mora ([2]) proved the following result:

(Shape lemma) *Let I be a radical 0-dimensional ideal in $\mathbb{Q}[z]$, regular in z_1. Then there are $g_1(z_1), \dots, g_n(z_1) \in \mathbb{Q}[z_1]$ such that g_1 is squarefree, $\deg(g_i) < \deg(g_1)$ for $i > 1$ and the Gröbner basis of the ideal I w.r.t. the lexicographical order $\succ$ with $z_1 \succ \cdots \succ z_n$ is of the form*

$$\{g_1(z_1), z_2 - g_2(z_1), \dots, z_n - g_n(z_1)\}. \tag{1.1}$$

On the other hand, if the reduced Gröbner basis of I w.r.t. $\succ$ is of this form, then I is a radical 0-dimensional ideal.

Furthermore, it is known that for "almost every" system of algebraic equations with finitely many solutions, after a suitable linear coordinate transformation, the reduced Gröbner basis of the transformed ideal will be in this simple form even though the system does not coincide with its radical ([5], [6], [7], [15]). The basis of the form (1.1) is called the shape basis of I.

We study the algebraic local cohomology class associated with the shape basis of a given 0-dimensional ideal I. We explicitly construct the holonomic system of linear partial differential equations for the algebraic local cohomology class. By making use of this holonomic system, we derive algorithms for computing point residues.

2 Notation and a former result

Let $X = \mathbb{C}^n$ and fix a coordinate system $z = (z_1, \dots, z_n)$ of X. We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X. Denote by $\mathcal{I}$ the zero dimensional ideal in $\mathcal{O}_X$ generated by holomorphic functions $f_1, \dots, f_n$ of z.

Put $Y = \{z \in X \mid f_1 = \cdots = f_n = 0\}$. The algebraic local cohomology group $\mathcal{H}^n_{[Y]}(\mathcal{O}_X)$ which satisfies $\mathcal{H}^n_{[Y]}(\mathcal{O}_X) = \lim\operatorname{ind}_k \mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^k, \mathcal{O}_X)$, has a structure of a left $\mathcal{D}_X$-module, where $\mathcal{D}_X$ is the sheaf of linear partial differential operators on

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X. Let $\left[\begin{array}{c} h \\ f_1\cdots f_n \end{array}\right]$ be a class in $\mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I},\mathcal{O}_X)$ for $h\in\mathcal{O}_X$. Denote by η the algebraic local cohomology class $\left[\dfrac{h}{f_1\cdots f_n}\right]$ defined by the image of $\left[\begin{array}{c} h \\ f_1\cdots f_n \end{array}\right]$ by the canonical mapping

$$\mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I},\mathcal{O}_X)\to\mathcal{H}^n_{[Y]}(\mathcal{O}_X). \tag{2.1}$$

Denote by $\mathcal{A}nn$ the ideal in $\mathcal{D}_X$ consisting of annihilators of η. Then we have $\mathcal{H}^n_{[Y]}(\mathcal{O}_X)\cong\mathcal{D}_X/\mathcal{A}nn$. For the Weyl algebra, it is possible to compute a Gröbner basis of $\mathcal{A}nn$ by using the computer algebra system Kan ([8], [9], [14]).

We have the canonical pairing

$$\begin{array}{rccl} \mathrm{Res}_\alpha: & \Omega_X\times\mathcal{H}^n_{[\alpha]}(\mathcal{O}_X) & \to & \mathbb{C} \\ & (\psi dz,\eta) & \mapsto & \mathrm{Res}_\alpha\langle\psi dz,\eta\rangle \end{array}$$

defined by the point residue $\mathrm{Res}_\alpha((h\psi)dz/f_1\cdots f_n)$ of a meromorphic differential form $(h\psi)dz/f_1\cdots f_n$ at $\alpha\in Y$.

The sheaf of holomorphic differential forms Ω_X is naturally endowed with a structure of a right $\mathcal{D}_X$-module by setting $(\phi(z)dz)R=((R^*\phi)(z))dz$ for a differential operator $R\in\mathcal{D}_X$, where R^* stands for the formal adjoint operator of R. Then we have, for any $R\in\mathcal{A}nn$,

$$\mathrm{Res}_\alpha\langle(R^*\phi(z))dz,\eta\rangle=\mathrm{Res}_\alpha\langle\phi(z)dz,R\eta\rangle=0,\ \alpha\in Y.$$

Theorem 2.1 ([10], [11]) *Put $\mathcal{K}=\{\phi(z)dz\in\Omega_X\mid Res_\alpha\langle\phi(z)dz,\eta\rangle=0,\forall\alpha\in Y\}$. Then we have*

$$\mathcal{K}=\{(R^*\psi(z))dz\mid R\in\mathcal{A}nn,\psi(z)dz\in\Omega_X\}.$$

3 Construction of the holonomic system in the shape basis case

Let us consider the system

$$(S)\begin{cases} f_1=g_1(z_1), \\ f_2=z_2-g_2(z_1), \\ \cdots\cdots\cdots, \\ f_n=z_n-g_n(z_1), \end{cases}$$

where $g_i(z_1)\in\mathbb{Q}[z_1]$. Denote by Y the set of common zeros of the system (S), i.e., $Y=\{z=(z_1,\dots,z_n)\in X\mid f_1=\cdots=f_n=0\}$. Put $\eta=[h/f_1\cdots f_n]\in\mathcal{H}^n_{[Y]}(\mathcal{O}_X)$ for $h\in\mathcal{O}_X$ with $h(\alpha)\neq0$, $\alpha\in Y$. Since η depends on the modulo class of h in $\mathcal{O}_X/\mathcal{I}$, the numerator h of the cohomology class η can be expressed as an univariate function of the variable z_1.

Let P, $F_1,\dots,F_n$ be differential operators defined by following forms:

$$(A)\begin{cases} P &=& \mathrm{sf}(g_1)\partial_1+\sum_{i=2}^n\mathrm{sf}(g_1)g_i'(z_1)\partial_i+\dfrac{g_1'(z_1)}{\gcd(g_1(z_1),g_1'(z_1))}-\dfrac{h'(z_1)}{h(z_1)}\mathrm{sf}(g_1), \\ F_1 &=& g_1(z_1), \\ F_2 &=& z_2-g_2(z_1), \\ \cdots & \cdots & \cdots, \\ F_n &=& z_n-g_n(z_1), \end{cases}$$

where $\mathrm{sf}(g_1)$ is the square free part $g_1(z_1)/\gcd(g_1(z_1),g_1'(z_1))$ of $g_1(z_1)$, $g_i'(z_1):=\partial g_i/\partial z_1$, and $\partial_i:=\partial/\partial z_i$, $i=1,\dots,n$. Then we have the next theorem.

Theorem 3.1 *Let $\mathcal{A}nn$ be the left ideal in $\mathcal{D}_X$ consisting of annihilators of η. Then $\mathcal{A}nn$ is generated by P and F_i, $i=1,\dots,n$ in (A).*

Proof. Recall the isomorphism

$$\mathcal{H}^n_{[Y]}(\mathcal{O}_X) \cong \frac{\mathcal{O}_X[*(Z_1 \cup \cdots \cup Z_n)]}{\sum_{i=1}^n \mathcal{O}_X[*(Z_1 \cup \cdots \cup \widehat{Z_i} \cup \cdots \cup Z_n)]}, \tag{3.1}$$

where $Z_i = \{z \in X \mid f_i(z) = 0\}$ and $\mathcal{O}_X[*Z]$ stands for a sheaf of meromorphic functions with poles at Z. By this isomorphism, we can readily see that operators in (A) annihilate η. Let $g_1 = \prod_{\iota=1}^{\nu}(z_1 - \alpha_{1,\iota})^{m_\iota}$ be the factorization of g_1 over $\mathbb{C}$. Then we have $\eta_\iota \in \mathcal{H}^n_{[\alpha_\iota]}(\mathcal{O}_X)$ such that $\eta = \eta_1 + \cdots + \eta_\nu$, where $\alpha_\iota = (\alpha_{1,\iota}, g_2(\alpha_{1,\iota}), \ldots, g_n(\alpha_{1,\iota})) \in Y$, $\iota = 1, \ldots, \nu$. Let U_k be a sufficiently small neighborhood of a point $\alpha_k \in Y$ and assume that $U_k \cap Y = \{\alpha_k\}$. Let us find the annihilators of η on U_k. Denote by $g_{i,k}$ the modulo class of g_i in $\mathcal{O}_X/\langle(z_1 - \alpha_{1,k})^{m_k}\rangle$. Put $f_{i,k}(z_1) = z_i - g_{i,k}(z_1)$. If we set $h_k = h/\prod_{\iota\neq k}(z_1 - \alpha_{1,\iota})^{m_\iota}$, we have

$$\eta_k = \left[\frac{h_k}{(z_1 - \alpha_{1,k})^{m_k} f_{2,k} \cdots f_{n,k}}\right].$$

Then we have

$$P_k = (z_1 - \alpha_{1,k})\partial_1 + (z_1 - \alpha_{1,k})\sum_{i\neq k} g'_{i,k}\partial_i + m_k - \frac{h'_k}{h_k}(z_1 - \alpha_{1,k}), \tag{3.2}$$

$$F_{1,k} = (z_1 - \alpha_{1,k})^{m_k}, \tag{3.3}$$

and

$$F_{i,k} = z_i - g_{i,k}(z_1),\ i = 2, \ldots, n \tag{3.4}$$

as annihilators of η on U_k. Note that the annihilator P_k can be rewritten as

$$P_k = (z_1 - \alpha_{1,k})\partial_1 + (z_1 - \alpha_{1,k})\sum_{i\neq k} g'_{i,k}\partial_i + \sum_{\iota=1}^{\nu} m_\iota \frac{z_1 - \alpha_{1,k}}{z_1 - \alpha_{1,\iota}} - \frac{h'}{h}(z_1 - \alpha_{1,k}). \tag{3.5}$$

We set $\mathcal{A}nn_k = \{R \in \mathcal{D}_X \mid R\eta_k = 0\}$. Since $\langle P_k, F_{1,k}, \ldots, F_{n,k}\rangle \subset \mathcal{A}nn_k$, we have a surjective morphism $\mathcal{D}_X/\langle P_k, F_{1,k}, \ldots, F_{n,k}\rangle \to \mathcal{D}_X/\mathcal{A}nn_k \to 0$. Recall that $\mathcal{D}_X/\mathcal{A}nn_k$ is a simple holonomic system, the multiplicity of $\mathcal{D}_X/\mathcal{A}nn_k$ is equal to 1. We can see that the multiplicity of $\mathcal{D}_X/\langle P_k, F_{1,k}, \ldots, F_{n,k}\rangle$ is also equal to 1. Thus $\mathcal{D}_X/\mathcal{A}nn_k = \mathcal{D}_X/\langle P_k, F_{1,k}, \ldots, F_{n,k}\rangle$ and finally we have $\langle P_k, F_{1,k}, \ldots, F_{n,k}\rangle = \mathcal{A}nn_k$. On the other hand, the localization of P and F_i, $i = 1, \ldots, n$ to U_k have the following forms:

$$\begin{aligned}
P|_{\alpha_k} &= \frac{1}{(z_1 - \alpha_{1,1})\ldots(z_1 - \alpha_{1,k-1})(z_1 - \alpha_{1,k+1})\ldots(z_1 - \alpha_{1,n})}P \\
&= (z_1 - \alpha_{1,k})\partial_1 + (z_1 - \alpha_{1,k})\sum_{i\neq k} g'_{i,k}\partial_i \\
&\quad + \sum_{\iota=1}^{\nu} m_\iota \frac{\prod_{\ell\neq\iota}(z_1 - \alpha_{1,\ell})}{(z_1 - \alpha_{1,1})\ldots(z_1 - \alpha_{1,k-1})(z_1 - \alpha_{1,k+1})\ldots(z_1 - \alpha_{1,n})} \\
&\quad - \frac{h'}{h}(z_1 - \alpha_{1,k}) \\
&= (z_1 - \alpha_{1,k})\partial_1 + (z_1 - \alpha_{1,k})\sum_{i\neq k} g'_{i,k}\partial_i + \sum_{\iota=1}^{\nu} m_\iota \frac{1}{z_1 - \alpha_{1,\iota}}(z_1 - \alpha_{1,k}) \\
&\quad - \frac{h'}{h}(z_1 - \alpha_{1,k}), && (3.6) \\
F_1|_{\alpha_k} &= (z_1 - \alpha_{1,k})^{m_k}, && (3.7) \\
F_i|_{\alpha_k} &= z_i - g_{i,k}(z_1),\ i = 2, \ldots, n. && (3.8)
\end{aligned}$$

According to the formulas from (3.3) to (3.8), we have $P|_{U_k} = P_k$, $F_i|_{U_k} = F_{i,k}$. Then we have $\mathcal{A}nn_k = \langle P|_{U_k}, F_1|_{U_k}, \dots, F_n|_{U_k}\rangle$. If we denote by $\mathcal{A}nn|_{U_k}$ the restriction of the ideal $\mathcal{A}nn$ to U_k, we have $\mathcal{A}nn|_{U_k} = \mathcal{A}nn_k$. Thus, we obtain that $\mathcal{A}nn|_{U_k} = \langle P|_{U_k}, F_1|_{U_k}, \dots, F_n|_{U_k}\rangle$. Consequently, $\mathcal{A}nn = \langle P, F_1, \dots, F_n\rangle$. □

3.1 Properties of P^*

The following relations between operators P and F_i, $i = 1, \dots, n$ hold:

Corollary 3.1

$$[P^*, F_i^*] = \begin{cases} -\dfrac{g_1'(z_1)}{\gcd(g_1(z_1), g_1'(z_1))} F_1, & i = 1, \\ 0, & i = 2, 3, \dots, n. \end{cases}$$

Proof. Since g_1 is a univariate polynomial of z_1, we have

$$\begin{aligned} [P^*, F_1^*] &= -\mathrm{sf}(g_1) \cdot g_i' \\ &= -\frac{g_1'}{\gcd(g_1, g_1')} F_1. \end{aligned}$$

For $i = 2, 3, \dots, n$, we have

$$[P^*, F_i^*] = -\mathrm{sf}(g_1) g_i' + \mathrm{sf}(g_1) g_i' = 0.$$

□

This corollary implies that, if $\varphi \in \mathcal{I}$, then $P^*\varphi \in \mathcal{I}$ holds. Thus, we have the next proposition.

Proposition 3.1 *P^* acts on the sheaf $\mathcal{O}_X/\mathcal{I}$, i.e.,*

$$P^* : \mathcal{O}_X/\mathcal{I} \to \mathcal{O}_X/\mathcal{I}.$$

Let $\tilde{\mathcal{I}}$ be the ideal generated by $\gcd(g_1(z_1), g_1'(z_1))$, $z_2 - g_2(z_1)$, $\dots$, $z_n - g_n(z_1)$ in $\mathcal{O}_X$. Then P^* has the following property:

Theorem 3.2 *A necessary and sufficient condition for $P^*\varphi(z) \in \mathcal{I}$ is $\varphi(z) \in \tilde{\mathcal{I}}$.*

Proof. We prove first that the condition is sufficient. Since $F_j^* = F_j = f_j$, we have $P^*(\chi f_i) = (P^*\chi) f_i$ for any $\chi \in \mathcal{O}_X$ by Corollary 3.1. Since the operator P^* can be written in the form

$$P^* = -\frac{g_1(z_1)}{h(z_1)} \partial_1 \frac{h(z_1)}{\gcd(g_1(z_1), g_1'(z_1))} - \sum_{i=2}^{n} \frac{g_1(z_1)}{\gcd(g_1(z_1), g_1'(z_1))} g_i'(z_1)\partial_i, \tag{3.9}$$

we have

$$\begin{aligned} P^*(\gcd(g_1, g_1')\varphi) = & -\frac{g_1}{h}\partial_1\Big(\frac{h}{\gcd(g_1, g_1')}\gcd(g_1, g_1')\varphi\Big) \\ & -\sum_{i=2}^{n} \frac{g_1}{\gcd(g_1, g_1')} g_i'\partial_i(\gcd(g_1, g_1')\varphi) \\ = & -\Big(\frac{1}{h}\partial_1 h\varphi + \sum_{i=2}^{n} g_1 g_i'\partial_i\varphi\Big) g_1 \\ = & -\Big(\frac{1}{h}\partial_1 h\varphi + \sum_{i=2}^{n} g_1 g_i'\partial_j\varphi\Big) f_1. \end{aligned}$$

These formulas imply the sufficiency. In order to prove the necessity, we set

$$\varphi(z) = \varphi_1(z)\gcd(g_1(z_1), g_1'(z_1)) + \varphi_2(z)f_2(z) + \cdots + \varphi_n(z)f_n(z) + \varphi_0(z_1),$$

where φ_0, φ_1, ..., $\varphi_n \in \mathcal{O}_X$ and φ_0 is an univariate polynomial of z_1 with $\deg\varphi_0(z_1) < \deg\gcd(g_1(z_1), g_1'(z_1))$. Since $P^*\varphi \in I$ by Corollary 3.1, there is an univariate polynomial $\psi(z_1)$ of z_1 such that $P^*\varphi_0(z_1) = \psi(z_1)f_1$. On the other hand, we have

$$P^*\varphi_0 = -\frac{g_1}{h}\partial_1\frac{h}{\gcd(g_1, g_1')}\varphi_0.$$

Thus we have

$$\begin{aligned}
-\frac{g_1}{h}\partial_1\frac{h}{\gcd(g_1,g_1')}\varphi_0 &= \psi f_1 \\
\frac{h}{\gcd(g_1,g_1')}\varphi_0 &= -\int^{z_1}\frac{h(t)}{g_1(t)}\psi(t)f_1(t)dt \\
\varphi_0 &= \left(-\frac{1}{h}\int^{z_1}\frac{h(t)}{g_1(t)}\psi(t)f_1(t)dt\right)\gcd(g_1,g_1').
\end{aligned}$$

Since $\varphi_0 \notin \tilde{\mathcal{I}}$, we have $\varphi_0 = 0$. This completes the proof. □

From the exact sequence $0 \to \tilde{\mathcal{I}}/\mathcal{I} \to \mathcal{O}_X/\mathcal{I} \to \mathcal{O}_X/\tilde{\mathcal{I}} \to 0$, we have that $\dim\Gamma(X, \tilde{\mathcal{I}}/\mathcal{I}) = \dim\Gamma(X, \mathcal{O}_X/\mathcal{I}) - \dim\Gamma(X, \mathcal{O}_X/\tilde{\mathcal{I}}) = \nu$. Put $d = \deg g_1(z_1)$. Then, we have the following corollary:

Corollary 3.2

(i) $\dim\Gamma(X, \mathrm{Im}(P^* : \mathcal{O}_X/\mathcal{I} \to \mathcal{O}_X/\mathcal{I})) = d - \nu$.

(ii) $\dim\Gamma(X, \mathrm{Ker}(P^* : \mathcal{O}_X/\mathcal{I} \to \mathcal{O}_X/\mathcal{I})) = \nu$.

Let $v_j(z_1)$ be the image of z_1^j by P^* in $\Gamma(X, \mathcal{O}_X/\mathcal{I})$ for $j = 0, \ldots, d-\nu-1$. Put $\mathcal{K} = \{v(z) \in \mathcal{O}_X \mid \mathrm{Res}_\alpha\langle v(z)dz, \eta\rangle = 0, \alpha \in Y\}$.

Corollary 3.3

$$\Gamma(X, \mathcal{K}/\mathcal{I}) \cong \mathrm{Span}\{v_0(z_1), \ldots, v_{d-\nu-1}(z_1)\}.$$

That is, any $v(z_1)$ which satisfies $\mathrm{Res}_\alpha\langle v(z_1)dz, \eta\rangle = 0$ for $\alpha \in Y$ and $\deg v(z_1) \leq d-1$ can be expressed as a linear combination of $v_0(z_1)$, ..., $v_{d-\nu-1}(z_1)$.

3.2 Localization

Let $g_1(z_1) = g_{1,1}^{\mu_1}(z_1)\cdots g_{1,N}^{\mu_N}(z_1)$ be the factorization of $g_1(z_1)$ over $\mathbb{Q}$. Let $g_{i,k}(z_1)$ be the remainder of division of $g_i(z_1)$ by $g_{1,k}^{\mu_k}(z_1)$. Put $f_{1,k}(z) = g_{1,k}^{\mu_k}(z_1)$ and $f_{i,k}(z) = z_i - g_{i,k}(z_1)$ for $k = 1, \ldots, N$ and $i = 2, \ldots, n$. Denote by I_k the ideal in $\mathbb{Q}[z]$ generated by $f_{1,k}(z)$, ..., $f_{n,k}(z)$. Let $F_{i,k}$ be the differential operator of order zero defined by $F_{i,k} = f_{i,k}$. From Corollary 3.1, we have the following formulas:

Corollary 3.4

$$[P^*, F_{i,k}^*] = \begin{cases} -((\prod_{j\neq i} g_{1,j})g_{1,k}')g_{1,k}^{\mu_k}, & i = 1, \\ 0, & i = 2, 3, \ldots, n. \end{cases} \tag{3.10}$$

These formulas imply the next result.

Lemma 3.1 *P^* acts on the vector space $\mathcal{O}_X/\mathcal{I}_k$, i.e.,*

$$P^* : \mathcal{O}_X/\mathcal{I}_k \to \mathcal{O}_X/\mathcal{I}_k.$$

Thus we can localize results in Section 3.1 to $\mathcal{I}_k$. Put $\nu_k = \deg g_{1,k}(z_1)$ and $d_k = \nu_k \mu_k$. Then we have the following:

Corollary 3.5

(i) $\dim \Gamma(X, \mathrm{Im}(P^* : \mathcal{O}_X/\mathcal{I}_k \to \mathcal{O}_X/\mathcal{I}_k)) = d_k - \nu_k$.

(ii) $\dim \Gamma(X, \mathrm{Ker}(P^* : \mathcal{O}_X/\mathcal{I}_k \to \mathcal{O}_X/\mathcal{I}_k)) = \nu_k$.

Let $v_{k,j}(z_1)$ be the image of z_1^j by P^* in $\Gamma(X, \mathcal{O}_X/\mathcal{I}_k)$ for $j = 0, \ldots, d_k - \nu_k - 1$. Denote by Y_k the set of common zeros of $f_{1,k}$, $\ldots$, $f_{n,k}$. Put $\mathcal{K}_k = \{v(z) \in \mathcal{O}_X \mid \mathrm{Res}_\alpha \langle v(z)dz, \eta_k \rangle = 0, \alpha \in Y_k\}$.

Corollary 3.6

$$\Gamma(X, \mathcal{K}_k/\mathcal{I}_k) \cong \mathrm{Span}\{v_{k,0}(z_1), \ldots, v_{k,d_k-\nu_k-1}(z_1)\}.$$

That is, any $v(z_1)$ which satisfies $\mathrm{Res}_{\alpha \in Y_k} \langle v(z_1)dz, \eta_k \rangle = 0$ and $\deg v(z_1) \leq d_k - 1$ can be expressed as a linear combination of $v_{k,0}(z_1)$, $\ldots$, $v_{k,d_k-\nu_k-1}(z_1)$.

4 Algorithm

We describe algorithms for computing point residues. Let $f_1(z)$, $\ldots$, $f_n(z)$ be polynomials in $\mathbb{Q}[z_1, \ldots, z_n]$ of the form (S) and $dz = dz_1 \wedge \cdots \wedge dz_n$. Let us consider a meromorphic differential form $\theta(z)dz/f_1(z)\cdots f_n(z)$ with a polynomial $\theta(z) \in \mathbb{Q}[z]$. Denote by $\underline{\theta}$ the remainder of θ by I. Now we introduce three vector spaces

$$U = \{u(z_1) \in \mathbb{Q}[z_1] \mid \deg u(z_1) \leq d - 1\}, \tag{4.1}$$

$$V = \{v(z_1) \in \mathbb{Q}[z_1] \mid \deg v(z_1) \leq d-1,\ \mathrm{Res}_\alpha \left(v(z_1)dz, \left[\frac{1}{f_1 \cdots f_n}\right] \right) = 0,\ \alpha \in Y\}, \tag{4.2}$$

and

$$W = \{w(z_1) \in \mathbb{Q}[z_1] \mid \deg w(z_1) \leq d - 1,\ \frac{w(z_1)}{f_1 \cdots f_n} \text{ has at most simple poles}\}. \tag{4.3}$$

The dimensions of these vector spaces are $\dim U = d$, $\dim V = d - \nu$ and $\dim W = \nu$, respectively. Let P be the annihilator of the cohomology class $[1/f_1 \cdots f_n]$ defined in (A), i.e.,

$$P = \mathrm{sf}(g_1)\partial_1 + \sum_{i=2}^{n} \mathrm{sf}(g_1) g_i'(z_1)\partial_i + \frac{g_1'(z_1)}{\gcd(g_1(z_1), g_1'(z_1))}.$$

Denote by $v_j(z_1)$ the remainder of $P^* z_1^j$ by $g_1(z)$, $j = 1, \ldots, d - \nu - 1$. Let Jac be Jacobian of f_1, $\ldots$, f_n. In this case, $\mathrm{Jac} = g_1'(z_1)$. Let $w_j(z_1)$ be the remainder of $\mathrm{Jac} \cdot z_1^\iota$ by $g_1(z_1)$ for $\iota = 0, \ldots, \nu - 1$.

Proposition 4.1

(i) $U = V \oplus W$

(ii) $V = \mathrm{Span}\{v_0(z_1), \ldots, v_{d-\nu-1}(z_1)\}$

(iii) $W = \mathrm{Span}\{w_0(z_1), \ldots, w_{\nu-1}(z_1)\}$

For computing the residues, we write

$$\underline{\theta}(z_1) = \sum_{j=0}^{d-\nu-1} a_j v_j(z_1) + \sum_{\ell=0}^{\nu-1} b_\ell w_\ell(z_1).$$

Then we have

$$\begin{aligned}\mathrm{Res}_{\alpha\in Y}\left(\frac{\theta(z_1)}{f_1\cdots f_n}dz\right) &= \mathrm{Res}_{\alpha\in Y}\left(\frac{\sum_{\ell=0}^{\nu-1} b_\ell w_\ell}{f_1\cdots f_n}dz\right)\\ &= \mathrm{Res}_{\alpha\in Y}\left(\left(\frac{\mathrm{Jac}}{f_1\cdots f_n}\sum_{\ell=0}^{\nu-1} b_\ell z_1^\ell\right)dz\right).\end{aligned}$$

Since $\mathrm{Jac}\sum_{\ell=0}^{\nu-1} b_\ell z_1^\ell dz/f_1\dots f_n$ is a meromorphic n-form with only simple poles, we can proceed as follows:

Let $g_1(z_1) = g_{1,1}^{\mu_1}(z_1)\cdots g_{1,N}^{\mu_N}(z_1)$ be the factorization of $g_1(z_1)$ over $\mathbb{Q}$. Denote by $g_{j,k}$ the remainder of g_j by $g_{1,k}^{\mu_k}$ and σ_k the remainder of $\sum_{\ell=0}^{\nu-1} b_\ell z_1^\ell$ by $g_{1,k}$. Let J_k be the ideal of $\mathbb{Q}[z,t]$ generated by $g_{1,k}$, $z_2 - g_{2,k}$, $\dots$, $z_n - g_{n,k}$ and $\mu_k\sigma_k - t$. We obtain a univariate polynomial $\varrho_k(t)$ of t as the generator of $J_k \cap \mathbb{Q}[t]$. Then $\varrho_k(t) = 0$ is the equation for residues of $\theta dz/f_1 \dots f_n$ at Y_k.

Algorithm 1 (point residues for shape basis case)

Input $g_1(z_1)$, $z_2 - g_2(z_1)$, $\dots$, $z_n - g_n(z_1)$ *: the shape basis,* $\theta(z) \in \mathbb{Q}[z]$
$\underline{\theta}(z_1) \leftarrow$ *the remainder of* $\theta(z)$ *by* $\langle g_1(z_1), z_2 - g_2(z_1), \dots, z_n - g_n(z_1)\rangle$
$\mathrm{sf}(g_1) \leftarrow g_1/\gcd(g_1, g_1')$
$\nu \leftarrow \deg \mathrm{sf}(g_1)$
$d \leftarrow \deg g_1$
for j **from** 0 **to** $d-\nu-1$
  $v_j \leftarrow$ *the remainder of* $-\dfrac{g_1}{\gcd(g_1,g_1')} j z_1^{j-1} + \dfrac{g_1}{\gcd(g_1,g_1')}\dfrac{\gcd(g_1,g_1')'}{\gcd(g_1,g_1')} z_1^j$ *by* $f_{1,k}$
for ℓ **from** 0 **to** $\nu - 1$
  $w_\ell \leftarrow$ *the remainder of* $g_1' z^\ell$ *of* g_1
$\vartheta \leftarrow \underline{\theta} - \sum_{j=0}^{d-\nu-1} a_j v_j - \sum_{\ell=0}^{\nu-1} b_\ell w_\ell$
$(a_0, \dots, a_{d-\nu-1}, b_0, \dots, b_{\nu-1}) \leftarrow$ *the coefficients s.t.* $\vartheta = 0$
$g_{1,1}^{\mu_1}\cdots g_{1,N}^{\mu_N} \leftarrow$ *the squarefree factorization of* g_1
for k **from** 1 **to** N
  for i **from** 2 **to** n
    $g_{i,k} \leftarrow$ *the remainder of* g_i *by* $g_{1,k}^{\mu_k}$
  $\sigma_k \leftarrow$ *the remainder of* $\sum_{\ell=0}^{\nu-1} b_\ell z^\ell$ *by* $g_{1,k}$
  $J_k \leftarrow \langle g_{1,k}, z_2 - g_{2,k}, \dots, z_n - g_{n,k}, \mu_k\sigma_k - t\rangle$
  $G_k \leftarrow$ *Gröbner basis of* J_k *w.r.t. the lexicographical order* $z \succ t$
Output $\{G_1, \dots, G_N\}$

Example 1 *Put* $z = (x,y)$. *Let us consider* $f_1 = x^4(2x^2-1)^3$, $f_2 = y - (x^3+1)$ *and* $\theta = 35xy^3 - x^2y + y - 1$. *The annihilator* P *of the cohomology class* $[1/f_1f_2]$ *is* $P = (2x^3 - x)\partial_x + (6x^5 - 3x^3)\partial_y + 20x^2 - 4$. *Then we have* $V = \mathrm{Span}\{v_0, \dots, v_6\}$, *where* $v_0 = 14x^2 - 3$, $v_1 = 12x^3 - 2x$, $v_2 = 10x^4 - x^2$, $v_3 = 8x^5$, $v_4 = 6x^6 + x^4$, $v_5 = 4x^7 + 2x^5$, $v_6 = 2x^8 + 3x^6$ *and* $W = \mathrm{Span}\{w_0, w_1, w_2\}$, *where*

$$\begin{aligned} w_0 &= 80x^9 - 96x^7 + 36x^5 - 4x^3,\\ w_1 &= 24x^8 - 24x^6 + 6x^4,\\ w_2 &= 24x^9 - 24x^7 + 6x^5.\end{aligned}$$

The remainder $\underline{\theta}$ *of* θ *by* $\langle f_1, f_2\rangle$ *is* $\underline{\theta} = (105/2)x^8 + 105x^7 - (105/4)x^6 - x^5 + (875/8)x^4 + x^3 - x^2 + 35x$. *It can be written as* $\underline{\theta} = -(35/2)v_1 + v_2 + (2425/16)v_3 +$

$(3555/64)v_4 - (739/4)v_5 - (1965/32)v_6 - (211/4)w_0 + (935/128)w_1 + (1055/6)w_2$. *Put* $\sigma = -(211/4) + (935/128)x + (1055/6)x^2$. *For* $I_1 = \langle x^4, y - (x^3+1)\rangle$, *we have* $\sigma_1 = -211/4$. *Thus,* $Res_{[(0,1)]}(\theta dz/f_1 f_2) = 4(-211/4) = -211$. *For* $I_2 = \langle (2x^2-1)^3, y-(x^3+1)\rangle$, *we have* $\sigma_2 = (935/128)x+(211/6)$. *Then* $G_2 = \langle -32768t^2 + 6914048t - 356848007, -2805x + 128t - 13504, -2805y + 64t - 3947\rangle$. *Thus we have* $\varrho_2(t) = -32768t^2 + 6914048t - 356848007 = 0$ *which* $t = \mathrm{Res}_{[V(I_2)]}(\theta dz/f_1 f_2)$ *satisfies.*

4.1 Localization

By using Corollary 3.6, we get an algorithm for computing the point residues of $\theta dz/f_1 \cdots f_n$. Let U_k, V_k and W_k be vector spaces given by

$$U_k := \{u(z_1) \in \mathbb{Q}[z_1] \mid \deg u(z_1) \le d_k - 1\},$$

$$V_k := \{v(z_1) \in \mathbb{Q}[z_1] \mid \deg v(z_1) \le d_k - 1,\ \mathrm{Res}_\alpha\left(\frac{v(z_1)}{f_{1,k}\cdots f_{n,k}}dz\right) = 0, \alpha \in Y_k\},$$

and

$$W_k = \{w(z_1) \in \mathbb{Q}[z_1] \mid \deg w(z_1) \le d_k - 1,\ \frac{w(z_1)}{f_{1,k}\cdots f_{n,k}} \text{ has at most simple poles}\}.$$

The dimensions of these spaces are $\dim U_k = d_k$, $\dim V_k = d_k - \nu_k$ and $\dim W_k = \nu_k$, respectively.

Denote by $v_{k,j}(z_1)$ the remainder of $P^* z_1^j$ by $f_{1,k}$, $j = 0, \ldots, d_k - \nu_k - 1$. Let $w_{k,j}(z_1)$ the remainder of $\mathrm{Jac} \cdot z_1^\ell$ by $f_{1,k}(z_1)$ for $\ell = 0, \ldots, \nu_k - 1$. Then we have the next proposition.

Proposition 4.2

(i) $U_k = V_k \oplus W_k$

(ii) $V_k = \mathrm{Span}\{v_{k,0}(z_1), \ldots, v_{k,d_k-\nu_k-1}(z_1)\}$

(iii) $W_k = \mathrm{Span}\{w_{k,0}(z_1), \ldots, w_{k,\nu_k-1}(z_1)\}$

Let $\underline{\theta}_k(z_1)$ be the remainder of $\underline{\theta}(z_1)$ by $f_{1,k}(z_1)$, where $\underline{\theta}(z_1)$ is the remainder of $\theta(z)$ by I. we can write $\underline{\theta}_k(z_1)$ into

$$\underline{\theta}_k(z_1) = \sum_{j=0}^{d_k-\nu_k-1} a_{k,j} v_{k,j}(z_1) + \sum_{\ell=0}^{\nu_k-1} b_{k,\ell} w_{k,\ell}(z_1)$$

and we have

$$\mathrm{Res}_{\alpha\in Y_k}\left(\frac{\theta}{f_1\cdots f_n}dz\right) = \mathrm{Res}_{\alpha\in Y_k}\left(\left(\frac{\mathrm{Jac}_k}{f_{1,k}\cdots f_{n,k}}\frac{\sum_{\ell=0}^{\nu_k-1} b_{k,\ell} z_1^\ell}{\prod_{j\neq k} f_{1,j}}\right)dz\right).$$

Thus we have that the residue of $\theta dz/f_1 \ldots f_n$ at $\alpha = (\alpha_1, \ldots, \alpha_n) \in Y_k$ is equal to $\mu_k(\sum_{\ell=0}^{\nu_k-1} b_{k,\ell}\alpha_1^\ell / \prod_{j\neq k} f_{1,j}(\alpha_1))$. In other words, for computing residues, we can proceed as follows:

Let J_k be the ideal of $\mathbb{Q}[z,t]$ generated by $f_{1,k}, f_{2,k}, \ldots, f_{n,k}$ and $\mu_k \sum_{\ell=0}^{\nu_k-1} b_{k,\ell} z_1^\ell - t\prod_{j\neq k} f_{1,j}$. We obtain an univariate polynomial $\varrho_k(t)$ of t as the generator of $J_k \cap \mathbb{Q}[t]$. Then $\varrho_k(t) = 0$ is the equation for residues of $\theta dz/f_1 \ldots f_n$ at Y_k.

Algorithm 2 (localized version)

Input $g_1(z_1),\ z_2 - g_2(z_1),\ \ldots,\ z_n - g_n(z_1)$ *: the shape basis,* $\theta(z) \in \mathbb{Q}[z]$
$\underline{\theta}(z_1) \leftarrow$ *the remainder of* $\theta(z)$ *by* $\langle g_1(z_1), z_2 - g_2(z_1), \ldots, z_n - g_n(z_1)\rangle$
$g_{1,1}^{\mu_1}(z_1) \cdots g_{1,N}^{\mu_N}(z_1) \leftarrow$ *the squarefree factorization of* $g_1(z_1)$
for k **from** 1 **to** N
  $f_{1,k} \leftarrow g_{1,k}^{\mu_k}$
  $\nu_k \leftarrow \deg g_{1,k}$
  $d_k \leftarrow \mu_k \cdot \nu_k$
  $\underline{\theta}_k \leftarrow$ *the remainder of* $\underline{\theta}$ *by* $f_{1,k}$
  for i **from** 2 **to** n
    $g_{i,k} \leftarrow$ *the remainder of* g_i *by* $f_{1,k}$
    $f_{i,k} \leftarrow z_i - g_{i,k}$
  for j **from** 0 **to** $d_k - \nu_k - 1$
    $v_{k,j} \leftarrow$ *the remainder of* $-\dfrac{g_1}{\gcd(g_1, g_1')} j z_1^{j-1} + \dfrac{g_1}{\gcd(g_1, g_1')} \dfrac{\gcd(g_1, g_1')'}{\gcd(g_1, g_1')} z_1^j$ *by* $f_{1,k}$
  for ℓ **from** 0 **to** $\nu_k - 1$
    $w_{k,\ell} \leftarrow$ *the remainder of* $f_{1,k}' z^\ell$ *by* $f_{1,k}$
  $\vartheta_k \leftarrow \underline{\theta}_k - \sum_{j=0}^{d_k-\nu_k-1} a_{k,j} v_{k,j} - \sum_{\ell=0}^{\nu_k-1} b_{k,\ell} w_{k,\ell}$
  $(a_{k,0}, \ldots, a_{k,d_k-\nu_k-1}, b_{k,0}, \ldots, b_{k,\nu_k-1}) \leftarrow$ *the coefficients s.t.* $\vartheta_k = 0$
  $J_k \leftarrow \langle g_{1,k}, f_{2,k}, \ldots, f_{n,k}, \gamma_k \sum_{\ell=0}^{\nu_k-1} b_\ell z_1^\ell - t \prod_{j \neq k} f_{1,k}\rangle$
  $G_k \leftarrow$ *Gröbner basis of* J_k *w.r.t. the lexicographic order* $z \succ t$
Output $\{G_1, \ldots, G_N\}$

Example 2 *Let us consider the same* f_1 *and* f_2 *with Example* 1. *For* $I_1 = \langle x^4, y - (x^3+1)\rangle$, $V_1 = \mathrm{Span}\{14x^2-3, 12x^3-2x, -x^2\}$ *and* $W_1 = \mathrm{Span}\{4x^3\}$. *The remainder* $\underline{\theta}_1$ *of* $\underline{\theta}$ *by* x^4 *is* x^3-x^2+35x. *It can be written as* $\underline{\theta}_1 = (211/4)w_{1,0} - (35/2)v_{1,1} + v_{1,2}$. *Thus we have* $G_1 = \langle t+211, x, y-1\rangle$. *In the same way, we have* $\underline{\theta}_2 = (211/24)w_{2,0} + (935/512)w_{2,1} + (375/256)v_{2,0} + (459/16)v_{2,1} + (1343/128)v_{2,2} - (513/16)v_{2,3}$. *Thus we have* $G_2 = \langle -32768t^2 + 6914048t - 356848007, -2805x + 128t - 13504, -2805y + 64t - 3947\rangle$ *for* $I_2 = \langle (2x^2-1)^3, y - (x^3+1)\rangle$.

Example 3 *Put* $z = (x, y)$. *Let us consider* $f_1 = (x^2+1)^{13}(2x^2-1)^9$ *and* $f_2 = y - (3x^6 + 3x^4 - 2x^3 + 2x^2 - 2x + 2)$. *The annihilator* P *of the cohomology class* $[1/f_1 f_2]$ *given in* (A) *is*
$P = (x^2+1)(2x^2-1)\partial_x + (36x^9 + 42x^7 - 12x^6 + 2x^5 - 10x^4 - 8x^3 + 4x^2 - 4x + 2)\partial_y + 88x^3 + 10x$.
Let us compute the residue of $\theta dz/f_1 f_2$, *where* $\theta = 35x^3y^5 - 2y^4 + 2xy - 1$. *Along the algorithm* 2, *for* $I_1 = \langle (2x^2-1)^9, y - (3x^6 + 3x^4 - 2x^3 + 2x^2 - 2x + 2)\rangle$, *we have that*

$$\mathrm{Res}_{[V(I_1)]}\left(\frac{\theta}{f_1 f_2} dz\right) = \mathrm{Res}_{[V(I_1)]}\left(\frac{\sigma_1}{f_{1,1} f_{1,2}} dz\right),$$

where

$$\sigma_1 = \left(-\frac{747718501}{5036466357} - \frac{126787493190876461}{380240477766549504} x\right) \mathrm{Jac}_1,$$

$\mathrm{Jac}_1 = 9216x^{17} - 36864x^{15} + 64512x^{13} - 64512x^{11} + 40320x^9 - 16128x^7 + 4032x^5 - 576x^3 + 36x$. *Thus we have*

$$\begin{aligned} G_1 = \ & \langle 4177342043388666896199639367 68t^2 - 1612447467044961048518379700224t \\ & -1778835134830001896609499526073, \\ & 1826154596005141x + 457019805007872t - 882044634267648, \\ & 14609236768041128y - 10968475320188928t - 39094030445746101\rangle. \end{aligned}$$

On the other hand, for $I_2 = \langle (x^2+1)^{13}, y - (3x^6 + 3x^4 - 2x^3 + 2x^2 - 2x + 2)\rangle$, *we have that* $\mathrm{Res}_{[V(I_2)]}(\theta dz/f_1 f_2)$ *satisfies*

$$\left\{\begin{array}{l}
85551965048599898034168614250086 4t^2 + 330229241250808022736564162605875 2t \\
+19261767639724614140497003918883305 = 0, \\
126787493190876461x + 29249267520503808t + 56450856593129472 = 0, \\
y = 0.
\end{array}\right.$$

We can apply our algorithms for any 0-dimensional ideal which has the shape basis even though the given generators are not the shape basis.

Example 4 *Let I be the ideal in $\mathbb{Q}[x,y]$ generated by $(x^2+y^2)^2+3x^2y-y^3$, x^2+y^2-1. Then I has the shape basis*

$$\{16x^6-24x^4+9x^2, y-(4x^4-5x^2+1)\}$$

with respect to the lexicographical order $y \succ x$. By the transformation law of the residue ([1]), we have

$$\mathrm{Res}_{\alpha\in Y}\left(\left[\frac{h}{((x^2+y^2)^2+3x^2y-y^3)(x^2+y^2-1)}\right]\right) = \mathrm{Res}_{\alpha\in Y}\left(\left[\frac{h\Delta}{f_1f_2}\right]\right)$$

for some $h \in \mathbb{Q}[x,y]$, where Y is the set of common zeros of $(x^2+y^2)^2+3x^2y-y^3$ and x^2+y^2-1 and $\Delta = -4x^2+1$.

Let us compute residues

$$\mathrm{Res}_{\alpha\in Y}\left(\left[\frac{h}{((x^2+y^2)^2+3x^2y-y^3)(x^2+y^2-1)}\right]\right)$$

for $h = 34x^5y+2x^3y^4-3x^2+42$. Put $\theta = h\Delta = -136yx^7+(-8y^4+34y)x^5+12x^4+2y^4x^3-171x^2+42$.

The annihilator P of the algebraic local cohomology class $[1/f_1f_2]$ given in (A) is

$$P = x(4x^2-3)\partial_x + (-8x^4+6x^2)\partial_y + 24x^2 - 6.$$

Put $I_1 = \langle x^2, y-1\rangle$ and $I_2 = \langle 16x^4-24x^2+9, y-x^2+5/4\rangle$. Then we have $\langle f_1, f_2\rangle = I_1 \cap I_2$.

For I_1, we have $v_{1,0} = -3$, $w_{1,0} = 2x$ and $\underline{\theta}_1 = -14v_{1,0}$. Thus the residue

$$\mathrm{Res}_{(0,0)}\left(\left[\frac{h}{((x^2+y^2)^2+3x^2y-y^3)(x^2+y^2-1)}\right]\right)$$

is equal to zero. On the other hand, for I_2, we have $v_{2,0} = 12x^2-3$, $v_{2,1} = 8x^3$, $w_{2,0} = 64x^3-48x$, $w_{2,1} = 48x^2-36$ and

$$\underline{\theta}_2 = \frac{213}{512}w_{2,0} + \frac{1}{8}w_{2,1} - \frac{53}{4}v_{2,0} + \frac{101}{32}v_{2,1}.$$

Thus we have $J_2 = \langle 2(213/512+(1/8)x)-t, 4x^2-3, 2y+1\rangle$ and

$$G_2 = \langle -12288t^2+27264t-14099, -64x+192t-213, 2y+1\rangle.$$

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